

Noncommutative gravity coupled to fermions: second order expansion via Seiberg-Witten map

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Abstract

We use the Seiberg-Witten map (SW map) to expand noncommutative gravity coupled to fermions in terms of ordinary commuting fields. The action is invariant under general coordinate transformations and local Lorentz rotations, and has the same degrees of freedom as the commutative gravity action. The expansion is given up to second order in the noncommutativity parameter θ .

A geometric reformulation and generalization of the SW map is presented that applies to any abelian twist. Compatibility of the map with hermiticity and charge conjugation conditions is proven. The action is shown to be real and invariant under charge conjugation at all orders in θ . This implies the bosonic part of the action to be even in θ , while the fermionic part is even in θ for Majorana fermions.

1 Introduction

Field theories defined on noncommutative spaces can be systematically constructed by use of an associative and noncommutative \star -product. This product between fields generates infinitely many derivatives and introduces a dimensionful noncommutativity parameter θ . The prototypical and simplest example of \star -product is the Groenewold-Moyal product [1] (historically arising in phase-space after Weyl quantization [2]) :

$$\begin{aligned} f(x) \star g(x) &\equiv \exp \left(\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right) f(x) g(y)|_{y \rightarrow x} \\ &= f(x) g(x) + \frac{i}{2} \theta^{\mu\nu} \partial_\mu f \partial_\nu g + \frac{1}{2!} \left(\frac{i}{2} \right)^2 \theta^{\mu_1 \nu_1} \theta^{\mu_2 \nu_2} (\partial_{\mu_1} \partial_{\mu_2} f) (\partial_{\nu_1} \partial_{\nu_2} g) + \dots \end{aligned} \quad (1.1)$$

with a constant θ .

Replacing in the classical theory the usual product by the star product leads to a deformation of the classical theory (called the noncommutative theory, or NC theory), containing an infinite number of new interactions and higher derivative terms. This procedure has been exploited to obtain deformed gravity theories in various dimensions [3, 4, 5, 6, 7, 8, 9, 10], including deformed supergravity [11, 12].

Such theories, seen as effective field theories, encode *ab initio* a noncommutative structure of spacetime, and it may be interesting to compare them to effective field theories emerging for example from string/brane interactions.

NC field theories are invariant under deformations of the classical symmetries: for instance the NC action for gauge fields is invariant under deformed gauge symmetries that involve \star -products.

The fields of NC theories can be expanded in a formal series in θ . Via the Seiberg-Witten (SW) map [13], this expansion can be realized in terms of classical fields (i.e. the original fields of the ordinary theory) transforming under the ordinary transformation laws [14, 15, 16]. In fact the SW map is explicitly determined by requiring that the NC transformation laws of the NC fields arise from the ordinary transformations of the classical fields, once the NC fields are seen as function of the classical fields. The SW map is a key ingredient in the construction of the NC standard model [17, 18], because it allows to have the same gauge group and degrees of freedom as in the commutative case.

In this paper we apply the Seiberg-Witten map to the NC vielbein gravity theory coupled to fermions developed in Ref. [10]. We determine the NC vielbein, spin connection, curvature and fermionic matter up to second order in θ , and expand the NC action to second order. We thus extend ordinary (commutative) vielbein gravity via higher order derivative terms dictated by NC vielbein gravity. All these results are given first for the usual Groenewold-Moyal product, and then for a general abelian twist determined by a set of commuting vector fields $\{X_I\}$.

The resulting action is a deformation of the commutative action, given in terms of the ordinary gravity fields and of the background noncommutativity vector fields $\{X_I\}$. All fields transform covariantly under ordinary diffeomorphisms, including the background fields $\{X_I\}$. These fields can be given dynamics by adding covariant kinetic and potential terms, in the spirit of ref. [19].

The action, being written geometrically, is invariant under general coordinate transformations.

Finally the expanded action is invariant under ordinary local Lorentz transformations (gauge transformations) of the classical fields, since these classical transformations induce, via the SW map, the NC transformations of the NC fields under which the NC action is invariant. Thus local Lorentz invariance is not broken in the θ -expanded NC action: in fact each order in θ of the expanded action is separately gauge invariant. This is not obvious at first sight because noncovariant terms, containing for example the “naked” spin connection, enter the expansion of the NC fields via the SW map. Only repeated integrations by parts allow to re-express the action in terms of gauge covariant quantities, where the spin connection appears in Lorentz covariant derivatives and curvatures.

The organization of the paper is as follows. In Section 2 we begin with a summary of NC vielbein gravity coupled to fermions and then prove the charge conjugation invariance of the action. In Section 3 the Seiberg-Witten map at all orders is recalled, and in Section 4 the map is found in the geometric language of exterior forms for a general abelian twist. In Section 5 all the fields of the NC theory are expressed via the SW map, up to second order in θ , in terms of the classical vielbein, spin connection, and Dirac fermions. In Section 6 we prove that the SW map is compatible with the hermiticity and charge conjugation conditions, ensuring that the action is real and even in θ . In Section 7 the action is expanded to second order. Section 8 contains our conclusions.

Appendix A is devoted to a short summary of twisted differential geometry, Appendix B deals with the hermiticity and charge conjugation of the SW map in a general setting. Appendix C contains gamma matrix conventions and properties.

2 Noncommutative vielbein gravity coupled to fermions

2.1 Classical action

The usual action of first-order gravity coupled to fermions can be recast in an index-free form, convenient for generalization to the non-commutative case:

$$S = \int Tr \left(iR \wedge V \wedge V \gamma_5 - [(D\psi)\bar{\psi} - \psi D\bar{\psi}] \wedge V \wedge V \wedge V \gamma_5 \right) \quad (2.1)$$

The fundamental fields are the 1-forms Ω (spin connection), V (vielbein) and the fermionic 0-form ψ (spin 1/2 field). The curvature 2-form R and the exterior covariant derivative on ψ and $\bar{\psi}$ are defined by

$$R = d\Omega - \Omega \wedge \Omega, \quad D\psi = d\psi - \Omega\psi, \quad D\bar{\psi} = d\bar{\psi} + \bar{\psi}\Omega \quad (2.2)$$

with

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab}, \quad V = V^a\gamma_a \quad (2.3)$$

and thus are 4×4 matrices in the spinor representation. See Appendix C for $D = 4$ gamma matrix conventions and useful relations. The Dirac conjugate is defined as usual: $\bar{\psi} = \psi^\dagger \gamma_0$.

Then also $(D\psi)\bar{\psi}$, $\psi D\bar{\psi}$ are matrices in the spinor representation, and the trace Tr is taken on this representation. Using the $D = 4$ gamma matrix identities:

$$\gamma_{abc} = i\varepsilon_{abcd}\gamma^d\gamma_5, \quad Tr(\gamma_{ab}\gamma_c\gamma_d\gamma_5) = -4i\varepsilon_{abcd} \quad (2.4)$$

leads to the usual action:

$$S = \int R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} + i[\bar{\psi}\gamma^a D\psi - (D\bar{\psi})\gamma^a\psi] \wedge V^b \wedge V^c \wedge V^d \varepsilon_{abcd} \quad (2.5)$$

with

$$R \equiv \frac{1}{4}R^{ab}\gamma_{ab}, \quad R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} \quad (2.6)$$

2.2 Invariances

The action is invariant under local diffeomorphisms (it is the integral of a 4-form on a 4-manifold) and under the local Lorentz rotations:

$$\delta_\epsilon V = -[V, \epsilon], \quad \delta_\epsilon \Omega = d\epsilon - [\Omega, \epsilon], \quad \delta_\epsilon \psi = \epsilon\psi, \quad \delta_\epsilon \bar{\psi} = -\bar{\psi}\epsilon \quad (2.7)$$

with

$$\epsilon = \frac{1}{4}\epsilon^{ab}\gamma_{ab} \quad (2.8)$$

The invariance can be directly checked on the action (2.1) noting that

$$\delta_\epsilon R = -[R, \epsilon] \quad \delta_\epsilon D\psi = \epsilon D\psi, \quad \delta_\epsilon((D\psi)\bar{\psi}) = -[(D\psi)\bar{\psi}, \epsilon], \quad \delta_\epsilon(\psi D\bar{\psi}) = -[\psi D\bar{\psi}, \epsilon], \quad (2.9)$$

using the cyclicity of the trace Tr (on spinor indices) and the fact that ϵ commutes with γ_5 . The Lorentz rotations close on the Lie algebra:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = -\delta_{[\epsilon_1, \epsilon_2]} \quad (2.10)$$

2.3 Hermiticity and charge conjugation

Since the vielbein V^a and the spin connection ω^{ab} are real fields, the following conditions hold:

$$\gamma_0 V \gamma_0 = V^\dagger, \quad -\gamma_0 \Omega \gamma_0 = \Omega^\dagger, \quad (2.11)$$

$$\gamma_0[(D\psi)\bar{\psi}]\gamma_0 = [\psi D\bar{\psi}]^\dagger, \quad \gamma_0[\psi D\bar{\psi}]\gamma_0 = [(D\psi)\bar{\psi}]^\dagger \quad (2.12)$$

and can be used to check that the action (2.1) is real.

Moreover, if C is the $D = 4$ charge conjugation matrix (antisymmetric and squaring to -1), we have

$$CVC = V^T, \quad C\Omega C = \Omega^T \quad (2.13)$$

since the matrices $C\gamma_a$ and $C\gamma_{ab}$ are symmetric.

Similar relations hold for the gauge parameter $\epsilon = (1/4)\varepsilon^{ab}\gamma_{ab}$:

$$-\gamma_0\epsilon\gamma_0 = \epsilon^\dagger, \quad C\epsilon C = \epsilon^T \quad (2.14)$$

ε^{ab} being real.

The charge conjugation of fermions:

$$\psi^C \equiv C(\bar{\psi})^T \quad (2.15)$$

can be extended to the bosonic fields V, Ω

$$V^C \equiv CV^TC, \quad \Omega^C \equiv C\Omega^TC \quad (2.16)$$

Then the relations (2.13) can be written as:

$$V^C = V, \quad \Omega^C = \Omega \quad (2.17)$$

and are the analogues of the Majorana condition for the fermions:

$$\psi^C = \psi \quad \rightarrow \quad \bar{\psi} = \psi^TC \quad (2.18)$$

So far we have been treating ψ as a Dirac fermion, and therefore reality of the action requires both terms in square brackets in the action (2.1) or (2.5). If ψ is Majorana, the two terms give the same contribution, and only one of them is necessary.

2.4 The noncommutative action and its invariances

After replacing exterior products by deformed exterior products (see Appendix A on twist differential geometry), the action(2.1) becomes:

$$S = \int Tr \left(iR \wedge_\star V \wedge_\star V \gamma_5 - [(D\psi) \star \bar{\psi} - \psi \star D\bar{\psi}] \wedge_\star V \wedge_\star V \wedge_\star V \gamma_5 \right) \quad (2.19)$$

with

$$R = d\Omega - \Omega \wedge_\star \Omega, \quad D\psi = d\psi - \Omega \star \psi \quad D\bar{\psi} = d\bar{\psi} + \bar{\psi} \star \Omega \quad (2.20)$$

Almost all preceding formulae continue to hold, with \star -products and \star -exterior products. However, the expansion of the fundamental fields on the Dirac basis of gamma matrices must now include new contributions¹:

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} + i\omega 1 + \tilde{\omega}\gamma_5, \quad V = V^a\gamma_a + \tilde{V}^a\gamma_a\gamma_5 \quad (2.21)$$

Similarly for the curvature :

$$R = \frac{1}{4}R^{ab}\gamma_{ab} + ir1 + \tilde{r}\gamma_5 \quad (2.22)$$

¹for example $\omega^{ab}\gamma_{ab} \wedge_\star \omega^{cd}\gamma_{cd} = \omega^{ab} \wedge_\star \omega^{cd}(-i\epsilon_{abcd}\gamma^5 - 4\delta_c^a\gamma_{bd} - 2\delta_{cd}^{ab}1)$ contains $1, \gamma_5$ besides γ_{ab} matrices since the \wedge_\star product is not antisymmetric.

and for the gauge parameter:

$$\epsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} + i\varepsilon 1 + \tilde{\varepsilon}\gamma_5 \quad (2.23)$$

Indeed now the \star -gauge variations read:

$$\delta_\epsilon V = -V \star \epsilon + \epsilon \star V, \quad \delta_\epsilon \Omega = d\epsilon - \Omega \star \epsilon + \epsilon \star \Omega, \quad \delta_\epsilon \psi = \epsilon \star \psi, \quad \delta_\epsilon \bar{\psi} = -\bar{\psi} \star \epsilon \quad (2.24)$$

and in the variations for V and Ω also anticommutators of gamma matrices appear, due to the noncommutativity of the \star -product. Since for example the anticommutator $\{\gamma_{ab}, \gamma_{cd}\}$ contains 1 and γ_5 , we see that the corresponding fields must be included in the expansion of Ω . Similarly, V must contain a $\gamma_a \gamma_5$ term due to $\{\gamma_{ab}, \gamma_c\}$. Finally, the composition law for gauge parameters becomes:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\epsilon_2 \star \epsilon_1 - \epsilon_1 \star \epsilon_2} \quad (2.25)$$

so that ϵ must contain the 1 and γ_5 terms, since they appear in the composite parameter $\epsilon_2 \star \epsilon_1 - \epsilon_1 \star \epsilon_2$.

The invariance of the noncommutative action (2.19) under the \star -variations is demonstrated in exactly the same way as for the commutative case, noting that

$$\delta_\epsilon R = -R \star \epsilon + \epsilon \star R, \quad \delta_\epsilon D\psi = \epsilon \star D\psi, \quad \delta_\epsilon ((D\psi) \star \bar{\psi}) = -(D\psi) \star \bar{\psi} \star \epsilon + \epsilon \star (D\psi) \star \bar{\psi} \quad (2.26)$$

etc., and using now the fact that ϵ still commutes with γ_5 , and the cyclicity of the trace Tr with respect to pointwise matrix products and the graded cyclicity of the integral with respect to the \star -product, so that $\int Tr$ is graded cyclic.

The local \star -symmetry satisfies the Lie algebra of $GL(2, C)$, and centrally extends the $SO(1, 3)$ Lie algebra of the commutative theory.

Finally, the \star -action (2.19) is invariant under diffeomorphisms generated by the Lie derivative, in the sense that

$$\int \mathcal{L}_v(4\text{-form}) = \int (i_v d + di_v)(4\text{-form}) = \int d(i_v(4\text{-form})) = \text{boundary term} \quad (2.27)$$

since $d(4\text{-form}) = 0$ on a 4-dimensional manifold. In fact the action is geometrical (it is the integral of a 4-form) and as such it is invariant under usual coordinate transformations.

2.5 Hermiticity and charge conjugation

Hermiticity conditions can be imposed on V , Ω and the gauge parameter ϵ :

$$\gamma_0 V \gamma_0 = V^\dagger, \quad -\gamma_0 \Omega \gamma_0 = \Omega^\dagger, \quad -\gamma_0 \epsilon \gamma_0 = \epsilon^\dagger \quad (2.28)$$

Moreover it is easy to verify the analogues of conditions (2.12):

$$\gamma_0 [(D\psi) \star \bar{\psi}] \gamma_0 = [\psi \star D\bar{\psi}]^\dagger, \quad \gamma_0 [\psi \star D\bar{\psi}] \gamma_0 = [D\psi \star \bar{\psi}]^\dagger \quad (2.29)$$

These hermiticity conditions are consistent with the gauge variations, as in the commutative case, and can be used to check that the action (2.19) is real. On the component fields V^a , \tilde{V}^a ,

ω^{ab} , ω , and $\tilde{\omega}$, and on the component gauge parameters ε^{ab} , ε , and $\tilde{\varepsilon}$ the hermiticity conditions (2.28) imply that they are real fields.

The charge conjugation relations (2.13), however, cannot be exported to the noncommutative case as they are. Indeed they would imply the vanishing of the component fields \tilde{V}^a , ω , and $\tilde{\omega}$ (whose presence is necessary in the noncommutative case) and anyhow would not be consistent with the \star -gauge variations.

An essential modification is needed, and makes use of the θ dependence of the noncommutative fields. This dependence will be made explicit in Section 3, using the Seiberg-Witten map. At this stage we just assume that there is such a dependence. Then we can impose consistent charge conjugation conditions as follows:

$$CV_\theta(x)C = V_{-\theta}(x)^T, \quad C\Omega_\theta(x)C = \Omega_{-\theta}(x)^T, \quad C\varepsilon_\theta(x)C = \varepsilon_{-\theta}(x)^T \quad (2.30)$$

These conditions can be checked to be consistent with the \star -gauge transformations. For example $CV_\theta(x)^TC$ can be shown to transform in the same way as $V_{-\theta}(x)$:

$$\begin{aligned} \delta_\epsilon(CV_\theta^TC) &= C(\delta_\epsilon V_\theta)^TC = C(-\epsilon_\theta^T \star_{-\theta} V_\theta^T + V_\theta^T \star_{-\theta} \epsilon_\theta^T)C = \\ &= \epsilon_{-\theta} \star_{-\theta} V_{-\theta} - V_{-\theta} \star_{-\theta} \epsilon_{-\theta} = \delta_\epsilon V_{-\theta} \end{aligned} \quad (2.31)$$

where we have used $C^2 = -1$ and the fact that the transposition of a \star -product of matrix-valued fields interchanges the order of the matrices but not of the \star -multiplied fields. To interchange both it is necessary to use the “reflected” $\star_{-\theta}$ product obtained by changing the sign of θ , since

$$f \star_\theta g = g \star_{-\theta} f \quad (2.32)$$

for any two functions f, g .

For the component fields and gauge parameters the charge conjugation conditions imply:

$$V_\theta^a = V_{-\theta}^a, \quad \omega_\theta^{ab} = \omega_{-\theta}^{ab} \quad (2.33)$$

$$\tilde{V}_\theta^a = -\tilde{V}_{-\theta}^a, \quad \omega_\theta = -\omega_{-\theta}, \quad \tilde{\omega}_\theta = -\tilde{\omega}_{-\theta}, \quad (2.34)$$

Similarly for the gauge parameters:

$$\varepsilon_\theta^{ab} = \varepsilon_{-\theta}^{ab} \quad (2.35)$$

$$\varepsilon_\theta = -\varepsilon_{-\theta}, \quad \tilde{\varepsilon}_\theta = -\tilde{\varepsilon}_{-\theta} \quad (2.36)$$

Finally, let us consider the charge conjugate spinor:

$$\psi^C \equiv C(\bar{\psi})^T \quad (2.37)$$

It transforms under \star -gauge variations as:

$$\delta_\epsilon \psi^C = C(\delta_\epsilon \bar{\psi})^T = C(-\bar{\psi} \star \epsilon)^T = C(-\epsilon^T \star_{-\theta} \psi^*) = C\epsilon^T C \star_{-\theta} C\psi^* = \epsilon_{-\theta} \star_{-\theta} \psi^C \quad (2.38)$$

i.e. it transforms in the same way as $\psi_{-\theta}$. Then we can impose the noncommutative Majorana condition:

$$\psi_\theta^C = \psi_{-\theta} \Rightarrow \psi_\theta^\dagger \gamma_0 = \psi_{-\theta}^T C \quad (2.39)$$

If the NC Majorana condition holds for ψ , it is immediate to verify that

$$C(D\psi_\theta \star_\theta \bar{\psi}_\theta)C = -(\psi \star D\bar{\psi})_{-\theta}^T \quad (2.40)$$

in close analogy with the charge conjugation conditions (2.30).

2.6 Reality and charge conjugation invariance of the action

Reality of the noncommutative action is proven by using the hermiticity conditions on V , Ω , R and on the fermion bilinears $D\psi \star \bar{\psi}$ and $\psi \star D\bar{\psi}$ when comparing the action (2.19) with its complex conjugate, obtained by taking the Hermitian conjugate of the 4-form in the overall trace inside the integral.

We define noncommutative charge conjugation to be the following transformation (extended linearly and multiplicatively to products of fields):

$$\psi \rightarrow \psi^C = C(\bar{\psi})^T = -\gamma_0 C \psi^*, \quad V \rightarrow V^C \equiv C V^T C, \quad \Omega \rightarrow \Omega^C \equiv C \Omega^T C, \quad \star_\theta \rightarrow \star_\theta^C = \star_{-\theta}, \quad (2.41)$$

and consequently $\wedge_{\star_\theta} \rightarrow \wedge_{\star_\theta^C} = \wedge_{\star_{-\theta}}$. Then the action (2.19) is invariant under charge conjugation. Indeed (setting for short $\wedge_{-\theta} \equiv \wedge_{\star_{-\theta}}$),

$$\begin{aligned} S^C &= \int Tr (iR^C \wedge_{-\theta} V^C \wedge_{-\theta} V^C \gamma_5 - [(D\psi^C) \star_{-\theta} \bar{\psi}^C - \psi^C \star_{-\theta} D\bar{\psi}^C] \wedge_{-\theta} V \wedge_{-\theta} V \wedge_{-\theta} V \gamma_5) \\ &= \int Tr (iR^C \wedge_{-\theta} V^C \wedge_{-\theta} V^C \gamma_5 - [(D\psi^C) \star_{-\theta} \bar{\psi}^C - \psi^C \star_{-\theta} D\bar{\psi}^C] \wedge_{-\theta} V \wedge_{-\theta} V \wedge_{-\theta} V \gamma_5)^T \\ &= S \end{aligned} \quad (2.42)$$

It may be useful to exhibit the various steps. Let us first concentrate on the bosonic part of the action. Then:

$$\begin{aligned} S_{bosonic}^C &= i \int Tr (R^C \wedge_{-\theta} V^C \wedge_{-\theta} V^C \gamma_5)^T = -i \int Tr (R^T \wedge_{-\theta} V^T \wedge_{-\theta} V^T C \gamma_5 C^{-1})^T \\ &= -i \int Tr ((V^T \wedge_{-\theta} V^T \gamma_5^T)^T \wedge_\star R) = -i \int Tr (-(V^T \gamma_5^T)^T \wedge_\star V \wedge_\star R) \\ &= i \int Tr (\gamma_5 V \wedge_\star V \wedge_\star R) = i \int Tr (R \wedge_\star \gamma_5 V \wedge_\star V) = i \int Tr (R \wedge_\star V \wedge_\star V \gamma_5) \\ &= S_{bosonic} \end{aligned} \quad (2.43)$$

Similarly the fermionic part of the action satisfies $S_{fermionic}^C = S_{fermionic}$. Let's first consider the connection terms in

$$\begin{aligned} S_{fermionic} &= \int -\bar{\psi} \star V \wedge_\star V \wedge_\star V \wedge_\star \gamma_5 d\psi - d\bar{\psi} \wedge_\star V \wedge_\star V \wedge_\star V \star \gamma_5 \psi \\ &\quad + \bar{\psi} \star V \wedge_\star V \wedge_\star V \wedge_\star \gamma_5 \Omega \star \psi - \bar{\psi} \star \Omega \wedge_\star V \wedge_\star V \wedge_\star V \star \gamma_5 \psi \end{aligned} \quad (2.44)$$

We find

$$\begin{aligned} (\bar{\psi} \star V \wedge_\star V \wedge_\star V \wedge_\star \gamma_5 \Omega \star \psi)^C &= \overline{\psi^C} \star_{-\theta} V^C \wedge_{-\theta} V^C \wedge_{-\theta} V^C \wedge_{-\theta} \gamma_5 \Omega^C \star_{-\theta} \psi^C \\ &= -(\bar{\psi})^T \star_{-\theta} V^T \wedge_{-\theta} V^T \wedge_{-\theta} V^T \wedge_{-\theta} \gamma_5^T \Omega^T \star (\bar{\psi})^T \\ &= \bar{\psi} \star \Omega \gamma_5 \wedge_\star V \wedge_\star V \wedge_\star V \star \psi \\ &= -\bar{\psi} \star \Omega \wedge_\star V \wedge_\star V \wedge_\star V \star \gamma_5 \psi \end{aligned} \quad (2.45)$$

where in the third line we inserted the definitions of the charge conjugate fields and simplified the C matrices by recalling that $\gamma_5^T = C\gamma_5 C^{-1}$; in the fourth line we transposed the whole expression (that is invariant because it is valued in complex numbers). We observe that the charge conjugation transformation squares to the identity; then the two connection terms in $S_{fermionic}$ are mapped one into the other under charge conjugation, and hence their sum is invariant.

The proof for the fermion kinetic term is similar, and can be obtained by replacing the spin connection with the exterior derivative, invariant under charge conjugation.

We can also conclude that the action *must be even in θ* if the fermions satisfy the Majorana condition. Indeed (2.30) and the Majorana fermions property (2.40) imply

$$V^C = V_{-\theta}, \quad \Omega^C = \Omega_{-\theta}, \quad \star_\theta^C = \star_{-\theta}, \quad [(D\psi) \star \bar{\psi} - \psi \star D\bar{\psi}]^C = [(D\psi) \star \bar{\psi} - \psi \star D\bar{\psi}]_{-\theta}. \quad (2.46)$$

Hence the bosonic action $S_{bosonic}(\theta)$ is mapped into $S_{bosonic}(-\theta)$ under charge conjugation. Also for the fermionic action $S_{fermionic}(\theta)$ we have $S_{fermionic}(\theta)^C = S_{fermionic}(-\theta)$ if the fermions are Majorana. Invariance of $S_{bosonic}$ and of $S_{fermionic}$ under charge conjugation then implies invariance of the action under $\theta \rightarrow -\theta$. Finally $S(\theta) = S(-\theta)$ implies that all corrections to the classical action are even in θ if we consider Majorana fermions.

2.7 Commutative limit $\theta \rightarrow 0$

In the commutative limit the action reduces to the usual action of gravity coupled to fermions of eq. (2.1). Indeed in virtue of the charge conjugation conditions on V and Ω , the component fields \tilde{V}^a , ω , and $\tilde{\omega}$ all vanish in the limit $\theta \rightarrow 0$ (see the second line of (2.34)), and only the classical spin connection ω^{ab} , vielbein V^a and Dirac fermion ψ survive. Similarly the gauge parameters ε , and $\tilde{\varepsilon}$ vanish in the commutative limit.

3 SW map for Groenewold-Moyal noncommutativity

In this section we consider Groenewold-Moyal noncommutativity, i.e. the star product is given by (1.1). The Seiberg-Witten map (SW map) relates the noncommutative gauge fields \hat{A} to the ordinary A , and the noncommutative gauge parameters $\hat{\varepsilon}$ to the ordinary ε and A so as to satisfy:

$$\hat{A}(A) + \delta_{\hat{\varepsilon}} \hat{A}(A) = \hat{A}(A + \delta_{\varepsilon} A) \quad (3.47)$$

with

$$\delta_{\varepsilon} A_{\mu} = \partial_{\mu} \varepsilon + i \varepsilon A_{\mu} - i A_{\mu} \varepsilon, \quad (3.48)$$

$$\delta_{\hat{\varepsilon}} \hat{A}_{\mu} = \partial_{\mu} \hat{\varepsilon} + i \hat{\varepsilon} \star \hat{A}_{\mu} - i \hat{A}_{\mu} \star \hat{\varepsilon}; \quad (3.49)$$

here, as usual in the literature on the subject, the A and ε transformations are chosen to be compatible with hermiticity (rather than anti-hermiticity) conditions.

The Seiberg-Witten condition (3.47) states that the dependence of the noncommutative gauge field on the ordinary one is fixed by requiring that ordinary gauge variations of A inside

$\hat{A}(A)$ produce the noncommutative gauge variation of \hat{A} . In a gauge theory physical quantities are gauge invariant: they do not depend on the gauge potential but on the equivalence class of potentials related by gauge transformations. The SW map relates the noncommutative gauge theory to the commutative one by requiring noncommutative fields to have the same gauge equivalence classes as the commutative ones. In this way the degrees of freedom of a noncommutative gauge theory are the same as those of the corresponding commutative one.

Equation (3.47) can be solved order by order in θ [16], yielding \hat{A} and $\hat{\varepsilon}$ as power series in θ :

$$\hat{A}(A, \theta) = A + A^1(A) + A^2(A) + \cdots + A^n(A) + \cdots \quad (3.50)$$

$$\hat{\varepsilon}(\varepsilon, A, \theta) = \varepsilon + \varepsilon^1(\varepsilon, A) + \varepsilon^2(\varepsilon, A) + \cdots + \varepsilon^n(\varepsilon, A) + \cdots \quad (3.51)$$

where $A^n(A)$ and $\varepsilon^n(\varepsilon, A)$ are of order n in θ . Note that $\hat{\varepsilon}$ depends on the ordinary ε and also on A .

In [13] it is shown that if \hat{A} and $\hat{\varepsilon}$ solve the differential equations

$$\frac{\partial}{\partial \theta^{\rho\sigma}} \hat{A}_\mu = -\frac{1}{4} \{ \hat{A}_{[\rho}, \partial_{\sigma]} \hat{A}_\mu + \hat{F}_{\sigma\mu} \}_\star \quad (3.52)$$

$$\frac{\partial}{\partial \theta^{\rho\sigma}} \hat{\varepsilon} = -\frac{1}{4} \{ \hat{A}_{[\rho}, \partial_{\sigma]} \hat{\varepsilon} \}_\star \quad (3.53)$$

with the definitions

$$\hat{F}_{\nu\rho} \equiv \partial_\nu \hat{A}_\rho - \partial_\rho \hat{A}_\nu - i \hat{A}_\nu \star \hat{A}_\rho + i \hat{A}_\rho \star \hat{A}_\nu \quad (3.54)$$

$$\{f, g\}_\star \equiv f \star g + g \star f \quad (3.55)$$

then \hat{A} and $\hat{\varepsilon}$ satisfy also the SW condition (3.47).

The differential equations (3.52), (3.53) admit solutions given recursively by [20]

$$A_\mu^{n+1} = -\frac{1}{4(n+1)} \theta^{\rho\sigma} \{ \hat{A}_\rho, \partial_\sigma \hat{A}_\mu + \hat{F}_{\sigma\mu} \}_\star^n \quad (3.56)$$

$$\varepsilon^{n+1} = -\frac{1}{4(n+1)} \theta^{\rho\sigma} \{ \hat{A}_\rho, \partial_\sigma \hat{\varepsilon} \}_\star^n \quad (3.57)$$

where $\{\hat{f}, \hat{g}\}_\star^n$ is n -th order term in $\{\hat{f}, \hat{g}\}_\star$, so that for example

$$\{ \hat{A}_\rho, \partial_\sigma \hat{\varepsilon} \}_\star^n \equiv \sum_{r+s+t=n} (A_\rho^r \star^s \partial_\sigma \varepsilon^t + \partial_\sigma \varepsilon^t \star^s A_\rho^r) \quad (3.58)$$

and \star^s indicates the s -th order term in the star product expansion (1.1). Here is a simple proof of (3.56), (3.57): multiplying the differential equations by $\theta^{\mu\nu}$ and analysing them order by order yields

$$\theta^{\mu\nu} \frac{\partial}{\partial \theta^{\mu\nu}} A_\rho^{n+1} = (n+1) A_\rho^{n+1} = -\frac{1}{4} \theta^{\mu\nu} \{ \hat{A}_{[\mu}, \partial_{\nu]} \hat{A}_\rho + \hat{F}_{\nu\rho} \}_\star^n \quad (3.59)$$

$$\theta^{\mu\nu} \frac{\partial}{\partial \theta^{\mu\nu}} \varepsilon^{n+1} = (n+1) \varepsilon^{n+1} = -\frac{1}{4} \theta^{\mu\nu} \{ \hat{A}_{[\mu}, \partial_{\nu]} \hat{\varepsilon} \}_\star^n \quad (3.60)$$

since A_ρ^{n+1} and ε^{n+1} are homogeneous functions of θ of order $n+1$.

Similar expressions hold for the gauge field strength, and for matter fields ϕ transforming in the fundamental or in the adjoint representation of the gauge group:

$$F_{\mu\nu}^{n+1} = -\frac{1}{4(n+1)}\theta^{\rho\sigma} \left(\{\hat{A}_\rho, \partial_\sigma \hat{F}_{\mu\nu} + D_\sigma \hat{F}_{\mu\nu}\}_\star^n - 2\{\hat{F}_{\mu\rho}, \hat{F}_{\nu\sigma}\}_\star^n \right) \quad (3.61)$$

$$\phi^{n+1} = -\frac{1}{4(n+1)}\theta^{\rho\sigma} \left(\hat{A}_\rho \star (\partial_\sigma \hat{\phi} + D_\sigma \hat{\phi}) \right)^n, \quad \delta_{\hat{\varepsilon}} \hat{\phi} = i\hat{\varepsilon} \star \hat{\phi} \quad (3.62)$$

$$\phi^{n+1} = -\frac{1}{4(n+1)}\theta^{\rho\sigma} \{\hat{A}_\rho, \partial_\sigma \hat{\phi} + D_\sigma \hat{\phi}\}_\star^n, \quad \delta_{\hat{\varepsilon}} \hat{\phi} = i\hat{\varepsilon} \star \hat{\phi} - i\hat{\phi} \star \hat{\varepsilon} \quad (3.63)$$

where the covariant derivative on \hat{F} and $\hat{\phi}$ is given by $D_\sigma \hat{F}_{\mu\nu} = \partial_\sigma \hat{F}_{\mu\nu} - i\hat{A}_\sigma \star \hat{F}_{\mu\nu} + i\hat{F}_{\mu\nu} \star \hat{A}_\sigma$, $D_\sigma \hat{\phi} = \partial_\sigma \hat{\phi} - i\hat{A}_\sigma \star \hat{\phi}$ (fundamental) and $D_\sigma \hat{\phi} = \partial_\sigma \hat{\phi} - i\hat{A}_\sigma \star \hat{\phi} + i\hat{\phi} \star \hat{A}_\sigma$ (adjoint).

Note The solution to (3.47) is not unique. For example if \hat{A} is a solution, any finite noncommutative gauge transformation of \hat{A} gives another solution. Another source of ambiguities is related to field redefinitions of the gauge potential. Both types of ambiguities should not lead to physical effects since the S matrix of the θ expanded theory (the noncommutative theory) is order by order in θ gauge invariant and is expected to be independent from field redefinitions. The ambiguities are anyhow constrained by two physical requirements:

- the hermiticity properties of the commutative fields must be extended to the noncommutative fields, and this implies reality of the NC actions;
- the charge conjugation properties of the commutative fields must be extended to the noncommutative fields, and this implies that commutative actions with charge conjugation symmetry can be deformed into noncommutative ones with noncommutative charge conjugation symmetry.

In this respect it is worth mentioning that the physically relevant ambiguities found up to second order in θ in [21] do not preserve the charge conjugation properties of the noncommutative fields [22].

It is therefore possible that the expansion of NC vielbein gravity to $\mathcal{O}(\theta^2)$ presented in the next Sections is unique up to physically irrelevant field redefinitions.

4 Geometric formulation of SW map for a general abelian twist

The SW map conditions can be formulated in the presence of an arbitrary (space-time dependent) NC star-product, not necessarily only in the Groenewold-Moyal case. The SW map with arbitrary noncommutativity has been constructed for abelian gauge theories in [26] using Kontsevich's results. In the nonabelian case the situation is more involved and there is no definite result (despite interesting partial ones [27]). In this section we show that when the

star-product is obtained via a set of mutually commuting vector fields (i.e. via an abelian twist) we can construct order by order solutions to the SW map. The mutually commuting vector fields can be spacetime dependent and hence we obtain a SW map for nonabelian gauge fields with nonconstant noncommutativity. The resulting NC gauge potential is then a 1-form that depends on the commutative gauge potential 1-form and on the mutually commutative vector fields.

The SW condition (3.47) can be seen as an 1-form equation, and therefore is coordinate independent. Likewise the differential eq.s (3.52),(3.53) can be recast in a coordinate independent form:

$$\frac{\partial}{\partial \theta^{IJ}} \hat{A} = -\frac{1}{4} \{i_{X_I} \hat{A}, \mathcal{L}_{X_J} \hat{A} + i_{X_J} \hat{F}\}_\star \quad (4.64)$$

$$\frac{\partial}{\partial \theta^{IJ}} \hat{\varepsilon} = -\frac{1}{4} \{i_{X_I} \hat{A}, \mathcal{L}_{X_J} \hat{\varepsilon}\}_\star \quad (4.65)$$

where all quantities and operations are diffeomorphic-convariant: \hat{A} is a one-form, $\hat{F} \equiv d\hat{A} - i\hat{A} \wedge_\star \hat{A}$ is a two-form, i_{X_I} and \mathcal{L}_{X_I} are respectively the contraction and the Lie derivative along the mutually commuting vector fields X_I . When the abelian twist reduces to the Moyal case of the preceding Section, the curvature becomes $\hat{F} = \frac{1}{2} \hat{F}_{\mu\nu} dx^\mu \wedge dx^\nu$, where the $\hat{F}_{\mu\nu}$ components are given in (3.54). Note that in the Moyal case $dx^\mu \wedge_\star dx^\nu = dx^\mu \wedge dx^\nu$ since $\mathcal{L}_{X_I} dx^\mu = d\mathcal{L}_{X_I} x^\mu = 0$.

Our strategy is the following: we first write down, in a coordinate independent way, candidate recursive solutions for the SW condition (3.47). Then we prove that in a particular coordinate system the candidate solutions satisfy the SW condition, and therefore must satisfy it in any coordinate system.

The candidate recursive solutions are given by:

$$A^{n+1} = -\frac{1}{4(n+1)} \theta^{IJ} \{i_{X_I} \hat{A}, \mathcal{L}_{X_J} \hat{A} + i_{X_J} \hat{F}\}_\star^n \quad (4.66)$$

$$\varepsilon^{n+1} = -\frac{1}{4(n+1)} \theta^{IJ} \{i_{X_I} \hat{A}, \mathcal{L}_{X_J} \hat{\varepsilon}\}_\star^n \quad (4.67)$$

We choose now a particular coordinate system adapted to the mutually commuting vectors X_I , i.e. precisely the coordinates y^I such that $X_I = \partial/\partial y^I$. It is then immediate to verify that the candidate solutions indeed reduce to the recursive solutions given in the preceding Section, where one just substitutes the coordinates x^μ with y^I . For example $\mathcal{L}_{X_I}(\hat{A}_J dy^J) = \partial \hat{A}_J / \partial y^I dy^J$, etc. Thus (4.66) and (4.67) are *bona fide* solutions of the SW equations in a generic coordinate system. The same argument can be used to prove that (4.66) and (4.67) are solutions of the differential equations (4.64) and (4.65), and that these latter imply the SW gauge condition (3.47).

Similarly one proves the generalization of eq.s (3.61)-(3.63):

$$F^{n+1} = -\frac{1}{4(n+1)} \theta^{IJ} \left(\{i_{X_I} \hat{A}, 2\mathcal{L}_{X_J} \hat{F} - i[i_{X_J} \hat{A}, \hat{F}]\}_\star^n - [i_{X_I} \hat{F}, i_{X_J} \hat{F}]_\star^n \right) \quad (4.68)$$

$$\phi^{n+1} = -\frac{1}{4(n+1)}\theta^{IJ}\left(i_{X_I}\hat{A}\star(2\mathcal{L}_{X_J}\hat{\phi}-i(i_{X_J}\hat{A})\star\hat{\phi})\right)^n, \quad \hat{\delta}_\varepsilon\hat{\phi}=i\varepsilon\star\hat{\phi} \quad (4.69)$$

$$\phi^{n+1} = -\frac{1}{4(n+1)}\theta^{IJ}\{i_{X_I}\hat{A}, 2\mathcal{L}_{X_J}\hat{\phi}-i(i_{X_J}\hat{A})\star\hat{\phi}+i\hat{\phi}\star(i_{X_J}\hat{A})\}_\star^n, \quad \hat{\delta}_\varepsilon\hat{\phi}=i\varepsilon\star\hat{\phi}-i\hat{\phi}\star\varepsilon \quad (4.70)$$

Note In order to prove that (4.66) and (4.67) satisfy the SW condition we have implicitly assumed that we have four commuting vector fields $\{X_I\}$ spanning (at each point of the four dimensional space-time manifold) the four dimensional tangent space-time. In this case we can indeed locally find a coordinate system $\{y^J\}$ such that $X_I = \partial/\partial y^I$ (Frobenius theorem). This assumption can be relaxed: (4.66) and (4.67) satisfy the SW condition also in the case of abelian twists $e^{-\frac{i}{2}\theta^{IJ}X_I\otimes X_J}$ where the N mutually commuting vector fields $\{X_I\}$ span a subspace of the four dimensional tangent space-time. The proof is algebraic and follows the same steps as the original proof given by Seiberg and Witten.

5 Expansion of fields to second order in θ

Here we apply the formulae of the preceding section to the gauge fields, matter fields and gauge parameters of noncommutative vielbein gravity coupled to fermions. The gauge field is the spin connection Ω , the matter fields are the vielbein (since it transforms in the adjoint representation of the gauge group, see (2.24)) and the fermi field (transforming in the fundamental representation). Comparison of the matter fields gauge transformations $D\psi = d\psi - \Omega \star \psi$ (defined in (2.20)) and $D_\sigma\hat{\phi} = \partial_\sigma\hat{\phi} - i\hat{A}_\sigma\star\hat{\phi}$ (defined below (3.63)), show that the formulae of the preceding section holds with $i\hat{A}$ replaced by Ω .

The solutions of the SW map provide explicit expressions for the fields in terms of θ and the ordinary fields. The expressions for V , Ω and R must have the all order gamma matrix structure:

$$V = V^a\gamma_a + \tilde{V}^1{}^a\gamma_a\gamma_5 + V^2{}^a\gamma_a + \tilde{V}^3{}^a\gamma_a\gamma_5 + \dots \quad (5.1)$$

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} + (i\omega^1 1 + \tilde{\omega}^1\gamma_5) + \frac{1}{4}\omega^2{}^{ab}\gamma_{ab} + (i\omega^3 1 + \tilde{\omega}^3\gamma_5) + \dots \quad (5.2)$$

$$R = \frac{1}{4}R^{ab}\gamma_{ab} + (iR^1 1 + \tilde{R}^1\gamma_5) + \frac{1}{4}R^2{}^{ab}\gamma_{ab} + (iR^3 1 + \tilde{R}^3\gamma_5) + \dots \quad (5.3)$$

(with all component fields $V^a, \tilde{V}^1{}^a, V^2{}^a, \tilde{V}^3{}^a, \omega^{ab}, \omega^1, \tilde{\omega}^1, etc.$ real) as can be deduced from (3.56), (3.63), (3.61). For example the first order term in Ω contains an anticommutator of γ_{ab} matrices, yielding 1 and γ_5 matrices. The second order term contains anticommutators of γ_{ab} with 1 and γ_5 , or commutators of γ_{ab} with γ_{ab} , yielding again γ_{ab} matrices, and so on. The gamma matrix structure depends therefore on the parity of the order in θ . The corrections up to second order in θ in the above expansions, and for the fermi field, are given in the following Tables, for Groenewold-Moyal twist, and for general abelian twist. The expansion up to first order appeared in [23] (see also [24]).

TABLE 1: SW fields at second order, Groenewold-Moyal product

Vielbein

$$V_\mu^{1\ a} = 0 \quad (5.4)$$

$$\tilde{V}_\mu^{1\ a} = \frac{1}{4}\theta^{\rho\sigma}\omega_\rho^{bc}\epsilon^{abcd}(\partial_\sigma V_\mu^d - \frac{1}{2}\omega_\sigma^{de}V_\mu^e) \quad (5.5)$$

$$\begin{aligned} V_\mu^{2\ a} = & -\frac{1}{8}\theta^{\rho\sigma}\left[4\omega_\rho^1(\partial_\sigma V_\mu^a - \frac{1}{2}\omega_\sigma^{ab}V_\mu^b) - \epsilon_{abcd}\omega_\rho^{bc}(\partial_\sigma \tilde{V}_\mu^d + \tilde{\omega}_\sigma^1 V_\mu^d - \frac{1}{2}\omega_\sigma^{de}\tilde{V}_\mu^e)\right] \\ & + \frac{1}{8}\theta^{\rho\sigma}\theta^{\nu\tau}\left[\omega_\rho^{bc}\partial_\nu\omega_\sigma^{bc}\partial_\tau V_\mu^a + 2\omega_\rho^{bc}\partial_\nu\omega_\sigma^{ca}\partial_\tau V_\mu^b - \partial_\nu\omega_\rho^{ab}\partial_\tau(\partial_\sigma V_\mu^b - \frac{1}{2}\omega_\sigma^{bc}V_\mu^c)\right] \end{aligned} \quad (5.6)$$

$$\tilde{V}_\mu^{2\ a} = 0 \quad (5.7)$$

Spin connection

$$\omega_\mu^{1\ ab} = 0 \quad (5.8)$$

$$\omega_\mu^1 = -\frac{1}{16}\theta^{\rho\sigma}\omega_\rho^{ab}(\partial_\sigma\omega_\mu^{ab} + R_{\sigma\mu}^{ab}) \quad (5.9)$$

$$\tilde{\omega}_\mu^1 = -\frac{1}{16}\theta^{\rho\sigma}\omega_\rho^{ab}(\partial_\sigma\omega_\mu^{cd} + R_{\sigma\mu}^{cd})\epsilon_{abcd} \quad (5.10)$$

$$\begin{aligned} \omega_\mu^{2\ ab} = & -\frac{1}{8}\theta^{\rho\sigma}(\partial_\sigma\omega_\mu^{cd} + R_{\sigma\mu}^{cd})(2\omega_\rho^1\delta_{cd}^{ab} + \tilde{\omega}_\rho^1\epsilon_{abcd}) \\ & - \frac{1}{4}\theta^{\rho\sigma}\omega_\rho^{cd}[(\partial_\sigma\omega_\mu^1 + R_{\sigma\mu}^1)\delta_{cd}^{ab} + \frac{1}{2}(\partial_\sigma\tilde{\omega}_\mu^1 + \tilde{R}_{\sigma\mu}^1)\epsilon_{abcd}] \\ & + \frac{1}{8}\theta^{\rho\sigma}\theta^{\nu\tau}(\partial_\nu\omega_\rho^{ac})\partial_\tau(\partial_\sigma\omega_\mu^{bc} + R_{\sigma\mu}^{bc}) \end{aligned} \quad (5.11)$$

$$\omega_\mu^2 = 0 \quad (5.12)$$

$$\tilde{\omega}_\mu^2 = 0 \quad (5.13)$$

Curvature

$$R_{\sigma\mu}^{1\ ab} = 0 \quad (5.14)$$

$$R_{\sigma\mu}^1 = -\frac{1}{16}\theta^{\nu\tau}[\omega_\nu^{ab}(\partial_\tau R_{\sigma\mu}^{ab} + D_\tau R_{\sigma\mu}^{ab}) - 2R_{\sigma\nu}^{ab}R_{\mu\tau}^{ab}] \quad (5.15)$$

$$\tilde{R}_{\sigma\mu}^1 = \frac{1}{32}\theta^{\nu\tau}[\omega_\nu^{ab}(\partial_\tau R_{\sigma\mu}^{cd} + D_\tau R_{\sigma\mu}^{cd})\epsilon_{abcd} - 2R_{\sigma\nu}^{ab}R_{\mu\tau}^{cd}\epsilon_{abcd}] \quad (5.16)$$

$$\begin{aligned} R_{\mu\nu}^{2\ ab} = & -\frac{1}{4}\theta^{\rho\sigma}[2\omega_\rho^{ab}\partial_\sigma R_{\mu\nu}^1 + 2\tilde{\omega}_\rho^{ab}\partial_\sigma \tilde{R}_{\mu\nu}^1 \\ & + \omega_\rho^1(\partial_\sigma R_{\mu\nu}^{ab} + D_\sigma R_{\mu\nu}^{ab}) + \tilde{\omega}_\rho^1(\partial_\sigma \tilde{R}_{\mu\nu}^{ab} + D_\sigma \tilde{R}_{\mu\nu}^{ab}) \\ & - 2R_{\mu\rho}^{ab}R_{\nu\sigma}^1 - 2\tilde{R}_{\mu\rho}^{ab}R_{\nu\sigma}^1 - 2R_{\mu\rho}^1 R_{\nu\sigma}^{ab} - 2\tilde{R}_{\mu\rho}^1 \tilde{R}_{\nu\sigma}^{ab}] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{32} \theta^{\rho\sigma} \theta^{\lambda\tau} [\tilde{\omega}_\lambda^{ab} \partial_\rho \tilde{\omega}_\tau^{cd} \partial_\sigma R_{\mu\nu}^{cd} - \omega_\lambda^{ab} \partial_\rho \omega_\tau^{cd} \partial_\sigma R_{\mu\nu}^{cd} \\
& - 4 \partial_\rho \omega_\lambda^{ca} \partial_\sigma (\partial_\tau R_{\mu\nu}^{cb} + D_\tau R_{\mu\nu}^{cb}) + 8 \partial_\rho R_{\mu\lambda}^{ca} \partial_\sigma R_{\nu\tau}^{cb}]
\end{aligned} \tag{5.17}$$

$$R_{\mu\nu}^2 = 0 \tag{5.18}$$

$$\tilde{R}_{\mu\nu}^2 = 0 \tag{5.19}$$

$$\text{with } \tilde{\omega}_\mu^{ab} \equiv \frac{1}{2} \epsilon^{abcd} \omega_\mu^{cd}, \tilde{R}_{\mu\nu}^{ab} \equiv \frac{1}{2} \epsilon^{abcd} R_{\mu\nu}^{cd}.$$

Fermion field

$$\psi^1 = \frac{1}{8} \theta^{\mu\nu} \omega_\mu^{ab} (\gamma_{ab} \partial_\nu \psi + \omega_\nu^{ac} \gamma_{bc} \psi) \tag{5.20}$$

$$\begin{aligned}
\psi^2 = & -\frac{1}{8} \theta^{\rho\sigma} [(\omega_\rho^1 - i\gamma_5 \tilde{\omega}_\rho^1)(\partial_\sigma \psi + D_\sigma \psi) - \frac{i}{4} \omega_\rho^{ab} \gamma_{ab} (\partial_\sigma \psi^1 + D_\sigma \psi^1) - (i\omega_\sigma^1 + \gamma_5 \tilde{\omega}_\sigma^1) \psi] \\
& + \frac{1}{64} \theta^{\rho\sigma} \theta^{\mu\nu} \gamma_{ab} [\frac{1}{4} \omega_\rho^{ab} \gamma_{cd} \partial_\mu \omega_\sigma^{cd} \partial_\nu \psi + \partial_\mu \omega_\rho^{ab} \partial_\nu (\partial_\sigma \psi + D_\sigma \psi)]
\end{aligned} \tag{5.21}$$

TABLE 2: SW forms at second order, general abelian twist

Vielbein

$$V^{1\ a} = 0 \tag{5.22}$$

$$\tilde{V}^{1\ a} = \frac{1}{4} \theta^{IJ} X_I^\rho \omega_\rho^{bc} \epsilon^{abcd} (\mathcal{L}_{X_J} V^d - \frac{1}{2} X_J^\sigma \omega_\sigma^{de} V^e) \tag{5.23}$$

$$\begin{aligned}
V^{2\ a} = & -\frac{1}{8} \theta^{IJ} X_I^\rho \left[4\omega_\rho^1 (\mathcal{L}_{X_J} V^a - \frac{1}{2} X_J^\sigma \omega_\sigma^{ab} V^b) - \epsilon_{abcd} \omega_\rho^{bc} (\mathcal{L}_{X_J} \tilde{V}^{1\ d} + X_J^\sigma \tilde{\omega}_\sigma^1 V^d - \frac{1}{2} X_J^\sigma \omega_\sigma^{de} \tilde{V}^{1\ e}) \right] \\
& + \frac{1}{8} \theta^{IJ} \theta^{KL} [X_I^\rho \omega_\rho^{bc} \mathcal{L}_{X_K} (X_J^\sigma \omega_\sigma^{bc}) \mathcal{L}_{X_L} V^a + 2\omega_\rho^{bc} \mathcal{L}_{X_K} (X_J^\sigma \omega_\sigma^{ca}) \mathcal{L}_{X_L} V^b \\
& - \mathcal{L}_{X_K} (X_I^\rho \omega_\rho^{ab}) \mathcal{L}_{X_L} (\mathcal{L}_{X_J} V^b - \frac{1}{2} X_J^\sigma \omega_\sigma^{bc} V^c)]
\end{aligned} \tag{5.24}$$

$$\tilde{V}^{2\ a} = 0 \tag{5.25}$$

Spin connection

$$\omega^{1\ ab} = 0 \tag{5.26}$$

$$\omega^1 = -\frac{1}{16} \theta^{IJ} X_I^\rho \omega_\rho^{ab} (\mathcal{L}_{X_J} \omega^{ab} + i_{X_J} R^{ab}) \tag{5.27}$$

$$\tilde{\omega}_\mu^1 = -\frac{1}{16} \theta^{IJ} X_I^\rho \omega_\rho^{ab} (\mathcal{L}_{X_J} \omega^{cd} + i_{X_J} R^{cd}) \epsilon_{abcd} \tag{5.28}$$

$$\begin{aligned}
\omega^{2\ ab} = & -\frac{1}{8} \theta^{IJ} X_I^\rho (\mathcal{L}_{X_J} \omega^{cd} + i_{X_J} R^{cd}) (2\omega_\rho^1 \delta_{cd}^{ab} + \tilde{\omega}_\rho^1 \epsilon_{abcd}) \\
& - \frac{1}{4} \theta^{IJ} X_I^\rho \omega^{cd} [(\mathcal{L}_{X_J} \omega^1 + i_{X_J} R^1) \delta_{cd}^{ab} + \frac{1}{2} ((\mathcal{L}_{X_J} \tilde{\omega}^1 + i_{X_J} \tilde{R}^1) \epsilon_{abcd})]
\end{aligned}$$

$$+ \frac{1}{8} \theta^{IJ} \theta^{KL} \mathcal{L}_{X_K} (X_I^\rho \omega_\rho^{ac}) \mathcal{L}_{X_L} (\mathcal{L}_{X_J} \omega^{bc} + i_{X_J} R^{bc}) \quad (5.29)$$

$$\omega^2 = 0 \quad (5.30)$$

$$\tilde{\omega}^2 = 0 \quad (5.31)$$

Curvature

$$R^{1\ ab} = 0 \quad (5.32)$$

$$R^1 = -\frac{1}{16} \theta^{IJ} [X_I^\nu \omega_\nu^{ab} (2\mathcal{L}_{X_J} R^{ab} - 2X_J^\tau \omega_\tau^{ac} R^{cb}) - 2(i_{X_I} R^{ab})(i_{X_J} R^{ab})] \quad (5.33)$$

$$\tilde{R}^1 = \frac{1}{32} \theta^{IJ} [X_I^\nu \omega_\nu^{ab} (2\mathcal{L}_{X_J} R^{cd} - 2X_J^\tau \omega_\tau^{ce} R^{ed}) \epsilon_{abcd} - 2(i_{X_I} R^{ab})(i_{X_J} R^{cd}) \epsilon_{abcd}] \quad (5.34)$$

$$\begin{aligned} R^{2\ ab} = & -\frac{1}{2} \theta^{IJ} [X_I^\rho \omega_\rho^{ab} \mathcal{L}_{X_J} R^1 + X_I^\rho \tilde{\omega}_\rho^{ab} \mathcal{L}_{X_J} \tilde{R}^1 \\ & + X_I^\rho \omega_\rho^1 (\mathcal{L}_{X_J} R^{ab} - X_J^\sigma \omega_\sigma^{ac} R^{cb}) + X_I^\rho \tilde{\omega}_\rho^1 (\mathcal{L}_{X_J} \tilde{R}^{ab} - X_J^\sigma \omega_\sigma^{ac} \tilde{R}^{cb}) \\ & - i_{X_I} (R^{ab} + \tilde{R}^{ab}) i_{X_J} R^1 - i_{X_I} (R^1 + \tilde{R}^1) i_{X_J} R^{ab}] \\ & + \frac{1}{32} \theta^{IJ} \theta^{KL} [X_K^\lambda \tilde{\omega}_\lambda^{ab} \mathcal{L}_{X_I} (X_L^\tau \tilde{\omega}_\tau^{cd}) \mathcal{L}_{X_J} R^{cd} - X_K^\lambda \omega_\lambda^{ab} \mathcal{L}_{X_I} (X_L^\tau \omega_\tau^{cd}) \mathcal{L}_{X_J} R^{cd} \\ & - 4\mathcal{L}_{X_I} (X_K^\lambda \omega_\lambda^{ca}) \mathcal{L}_{X_J} (2\mathcal{L}_{X_L} R^{cb} - 2X_L^\tau \omega_\tau^{cd} R^{db}) + 8\mathcal{L}_{X_I} (i_{X_K} R^{ca}) \mathcal{L}_{X_J} (i_{X_L} R^{cb})] \end{aligned} \quad (5.35)$$

$$R^2 = 0 \quad (5.36)$$

$$\tilde{R}^2 = 0 \quad (5.37)$$

$$\text{with } \tilde{\omega}^{ab} \equiv \frac{1}{2} \epsilon^{abcd} \omega^{cd}, \tilde{R}^{ab} \equiv \frac{1}{2} \epsilon^{abcd} R^{cd}.$$

Fermion field

$$\psi^1 = \frac{1}{8} \theta^{IJ} X_I^\mu \omega_\mu^{ab} (\gamma_{ab} \mathcal{L}_{X_J} \psi + X_J^\nu \omega_\nu^{ac} \gamma_{bc} \psi) \quad (5.38)$$

$$\begin{aligned} \psi^2 = & -\frac{1}{8} \theta^{IJ} X_I^\rho \left[(\omega_\rho^1 - i\gamma_5 \tilde{\omega}_\rho^1) (2\mathcal{L}_{X_J} \psi - \frac{1}{4} X_J^\sigma \omega_\sigma^{cd} \gamma_{cd} \psi) \right. \\ & \left. - \frac{i}{4} X_I^\rho \omega_\rho^{ab} \gamma_{ab} (2\mathcal{L}_{X_J} \psi^1 - \frac{1}{4} X_J^\sigma \omega_\sigma^{cd} \gamma_{cd} \psi^1) - X_J^\sigma (i\omega_\sigma^1 + \gamma_5 \tilde{\omega}_\sigma^1) \psi \right] \\ & + \frac{1}{64} \theta^{IJ} \theta^{KL} \gamma_{ab} \left[\frac{1}{4} X_I^\rho \omega_\rho^{ab} \gamma_{cd} \mathcal{L}_{X_K} (X_J^\sigma \omega_\sigma^{cd}) \mathcal{L}_{X_L} \psi + \mathcal{L}_{X_K} (X_I^\rho \omega_\rho^{ab}) \mathcal{L}_{X_L} (2\mathcal{L}_{X_J} \psi - \frac{1}{4} X_J^\sigma \omega_\sigma^{cd} \gamma_{cd} \psi) \right] \end{aligned} \quad (5.39)$$

6 Compatibility of SW map and NC gravity action with the hermiticity and charge conjugation conditions

Since the hermiticity properties (2.28) are essential to ensure reality of the NC action, it is necessary to check whether the SW solutions indeed satisfy these properties. This can be seen

directly on the SW solutions (5.1) - (5.3), and is due to their gamma matrix structure (odd gamma matrices for V , even gamma matrices for Ω and R) and to the reality of the component fields.

Similarly one can argue for the charge conjugation conditions (2.30): the matrix structure of (5.1) - (5.3) indeed is such that these conditions hold, because even terms in θ multiply symmetric gamma matrices (in the sense that $C\gamma$ is symmetric), while odd terms multiply antisymmetric gamma matrices.

The compatibility between the SW map and the hermiticity and charge conjugation conditions on all the fields is proven to all orders in θ and for a general field representation in Appendix B.

When the noncommutative fields are expressed in terms of the commutative ones via the SW map, these latter are considered to be the elementary dynamical fields. The charge conjugation operation is then the usual operation on commutative fields, and is extended to the noncommutativity parameter θ via the rule

$$\theta \rightarrow \theta^C = -\theta \quad (6.1)$$

corresponding to $\star_\theta \rightarrow \star_\theta^C = \star_{-\theta}$, cf. (2.41). The compatibility of SW map with the charge conjugation condition reads (see also (B.36) and (B.38)):

$$\hat{\psi} \rightarrow \hat{\psi}^C = -\gamma_0 C \hat{\psi}^*, \quad \hat{V} \rightarrow \hat{V}^C = C \hat{V}^T C, \quad \hat{\Omega} \rightarrow \hat{\Omega}^C = C \hat{\Omega}^T C, \quad (6.2)$$

where

$$\hat{\psi} = \hat{\psi}(\psi, \Omega, \theta), \quad \hat{V} = \hat{V}(V, \Omega, \theta), \quad \hat{\Omega} = \hat{\Omega}(\Omega, \theta).$$

and

$$\hat{\psi}^C = \hat{\psi}(\psi^C, \Omega^C, \theta^C), \quad \hat{V}^C = \hat{V}(V^C, \Omega^C, \theta^C), \quad \hat{\Omega}^C = \hat{\Omega}(\Omega^C, \theta^C) \quad (6.3)$$

Since the transformations (6.2) are the same as (2.41), they immediately imply that the noncommutative gravity action coupled to spinor fields and expanded via SW map to any order in the noncommutativity parameter θ in terms of the commutative field, is charge conjugation invariant. As in Section 2.6 this implies that the bosonic action is even in θ and that the fermionic part is also even in θ if it describes a Majorana fermion coupled to gravity.

In particular, the bosonic action must vanish at first order in θ , as we verify explicitly in the next Section.

7 The noncommutative action expanded to second order in θ

7.1 The bosonic action at first order vanishes

This can be explicitly verified: let us consider the first order pure gravity action:

$$S^1 \equiv \int Tr (iR \wedge_\star V \wedge_\star V \gamma_5)^1 = \int Tr (iR^1 \wedge V^0 \wedge V^0 \gamma_5 + iR^0 \wedge (V \wedge_\star V \gamma_5)^1) \quad (7.1)$$

where

$$\begin{aligned}
(V \wedge_\star V \gamma_5)^1 &= V^1 \wedge V^0 \gamma_5 + V^0 \wedge V^1 \gamma_5 + \frac{i}{2} \theta^{\rho\sigma} \partial_\rho V_\mu^0 \partial_\sigma V_\nu^0 \gamma_5 dx^\mu \wedge dx^\nu = \\
&= \tilde{V}^1{}^a \wedge V^b \gamma_a \gamma_5 \gamma_b + V^b \wedge \tilde{V}^1{}^a \gamma_b \gamma_a \gamma_5 + \frac{i}{2} \theta^{\rho\sigma} \partial_\rho V_\mu^a \partial_\sigma V_\nu^b \gamma_a \gamma_b \gamma_5 dx^\mu \wedge dx^\nu \\
&= -2\tilde{V}^1{}^a \wedge V^a \gamma_5 + i\theta^{\rho\sigma} \partial_\rho V_\mu^a \partial_\sigma V_\nu^a \gamma_5 dx^\mu \wedge dx^\nu
\end{aligned} \tag{7.2}$$

and we have used that up to boundary terms:

$$\int Tr(iR \wedge_\star V \wedge_\star V \gamma_5) = \int Tr(iR \wedge (V \wedge_\star V \gamma_5)) \tag{7.3}$$

Recalling that R^1 has only 1 and γ_5 parts, and R^0 has only the γ_{ab} part, the trace in (7.1) is only over γ_{ab} or $\gamma_{ab}\gamma_5$ yielding always 0. We have thus verified that the first order part of the pure gravity NC action vanishes.

7.2 The action at second order

The second order part of the NC action reads:

$$\begin{aligned}
S^2 &\equiv \int Tr(iR \wedge_\star V \wedge_\star V \gamma_5 - (D\psi \star \bar{\psi} - \psi \star D\bar{\psi}) \wedge_\star V \wedge_\star V \wedge_\star V \gamma_5)^2 = \\
&= \int Tr[iR^2 \wedge V^0 \wedge V^0 \gamma_5 + R^0 \wedge (V \wedge_\star V \gamma_5)^2 + R^1 \wedge (V \wedge_\star V \gamma_5)^1 \\
&\quad - (D\psi \star \bar{\psi} - \psi \star D\bar{\psi})^2 \wedge V \wedge V \wedge V \gamma_5 - (D\psi \star \bar{\psi} - \psi \star D\bar{\psi})^0 \wedge (V \wedge_\star V \wedge_\star V \gamma_5)^2 \\
&\quad - (D\psi \star \bar{\psi} - \psi \star D\bar{\psi})^1 \wedge (V \wedge_\star V \wedge_\star V \gamma_5)^1]
\end{aligned} \tag{7.4}$$

Expanding the fields as in preceding Section, and carrying out the traces yields:

$$\begin{aligned}
S^2 &= \int (R^{2ab} \wedge V^c \wedge V^d + 2R^{ab} \wedge V^{2c} \wedge V^d - 2R^{ab} \wedge V^{1c} \wedge V^{1d}) \epsilon_{abcd} \\
&\quad - 2\theta^{IJ} R^{ab} \wedge \mathcal{L}_{X_I} V^{1a} \wedge \mathcal{L}_{X_J} V^b + 8R^1 \wedge V^{1a} \wedge V^a - 4\theta^{IJ} \tilde{R}^1 \wedge \mathcal{L}_{X_I} V^a \wedge \mathcal{L}_{X_J} V^a
\end{aligned} \tag{7.5}$$

for the pure gravity part, and

$$\begin{aligned}
S_\psi^2 &= \int [\bar{\psi}^2 \gamma_{abc} \gamma_5 D\psi + \bar{\psi} \gamma_{abc} \gamma_5 (D\psi)^2 + \bar{\psi}^1 \gamma_{abc} \gamma_5 (D\psi)^1 \\
&\quad + \frac{i}{2} \theta^{IJ} (\mathcal{L}_{X_I} \bar{\psi}^1 \gamma_{abc} \gamma_5 \mathcal{L}_{X_J} D\psi + \mathcal{L}_{X_I} \bar{\psi} \gamma_{abc} \gamma_5 \mathcal{L}_{X_J} (D\psi)^1) \\
&\quad + \frac{1}{8} \theta^{IJ} \theta^{KL} ((\mathcal{L}_{X_I} \mathcal{L}_{X_K} \bar{\psi}^1) \gamma_{abc} \gamma_5 \mathcal{L}_{X_J} \mathcal{L}_{X_L} D\psi + (\mathcal{L}_{X_I} \mathcal{L}_{X_K} \bar{\psi}) \gamma_{abc} \gamma_5 \mathcal{L}_{X_J} \mathcal{L}_{X_L} (D\psi)^1)] \\
&\quad \wedge V^a \wedge V^b \wedge V^c \\
&\quad + [\bar{\psi}^1 \gamma_a \gamma_b \gamma_c D\psi + \bar{\psi} \gamma_a \gamma_b \gamma_c (D\psi)^1 + \frac{i}{2} \theta^{IJ} \mathcal{L}_{X_I} \bar{\psi} \gamma_a \gamma_b \gamma_c \mathcal{L}_{X_J} D\psi] \wedge
\end{aligned}$$

$$\begin{aligned}
& \wedge (V^{1\ a} \wedge V^b \wedge V^c - V^a \wedge V^{1\ b} \wedge V^c + V^a \wedge V^b \wedge V^{1\ c}) \\
& + \frac{i}{2} \theta^{IJ} [\bar{\psi}^1 \gamma_a \gamma_b \gamma_c \gamma_5 D\psi + \bar{\psi} \gamma_a \gamma_b \gamma_c \gamma_5 (D\psi)^1 + \frac{i}{2} \theta^{KL} \mathcal{L}_{X_K} \bar{\psi} \gamma_a \gamma_b \gamma_c \gamma_5 \mathcal{L}_{X_L} D\psi] \wedge \\
& \wedge (\mathcal{L}_{X_I} V^a \wedge \mathcal{L}_{X_J} V^b \wedge V^c + V^a \wedge \mathcal{L}_{X_I} V^b \wedge \mathcal{L}_{X_J} V^c + \mathcal{L}_{X_I} V^a \wedge V^b \wedge \mathcal{L}_{X_J} V^c) \\
& + \bar{\psi} \gamma_a \gamma_b \gamma_c D\psi \wedge K^{2\ abc} + \bar{\psi} \gamma_a \gamma_b \gamma_c \gamma_5 D\psi \wedge L^{2\ abc} \\
& - (\psi \leftrightarrow D\psi)
\end{aligned} \tag{7.6}$$

for the fermi field part, where $K^{2\ abc}$ and $L^{2\ abc}$ are vielbein combinations originating from $(V \wedge_\star V \wedge_\star V \gamma_5)^2$:

$$\begin{aligned}
K^{2\ abc} \equiv & \frac{i}{2} \theta^{IJ} (\mathcal{L}_{X_I} V^{1\ a} \wedge \mathcal{L}_{X_J} V^b \wedge V^c + \mathcal{L}_{X_I} V^{1\ a} \wedge V^b \wedge \mathcal{L}_{X_J} V^c + V^{1\ a} \wedge \mathcal{L}_{X_I} V^b \wedge \mathcal{L}_{X_J} V^c \\
& + \mathcal{L}_{X_I} V^a \wedge \mathcal{L}_{X_J} V^{1\ b} \wedge V^c + \mathcal{L}_{X_I} V^a \wedge V^{1\ b} \wedge \mathcal{L}_{X_J} V^c + V^a \wedge \mathcal{L}_{X_I} V^{1\ b} \wedge \mathcal{L}_{X_J} V^c \\
& + \mathcal{L}_{X_I} V^a \wedge \mathcal{L}_{X_J} V^b \wedge V^{1\ c} + \mathcal{L}_{X_I} V^a \wedge V^b \wedge \mathcal{L}_{X_J} V^{1\ c} + V^a \wedge \mathcal{L}_{X_I} V^b \wedge \mathcal{L}_{X_J} V^{1\ c})
\end{aligned} \tag{7.7}$$

$$\begin{aligned}
L^{2\ abc} \equiv & V^{2\ a} \wedge V^b \wedge V^c - V^{1\ a} \wedge V^{1\ b} \wedge V^c + V^a \wedge V^{2\ b} \wedge V^c \\
& + V^a \wedge V^b \wedge V^{2\ c} + V^{1\ a} \wedge V^b \wedge V^{1\ c} - V^a \wedge V^{1\ b} \wedge V^{1\ c} \\
& - \frac{1}{4} \theta^{IJ} \theta^{KL} (\mathcal{L}_{X_I} \mathcal{L}_{X_K} V^a \wedge \mathcal{L}_{X_L} V^b \wedge \mathcal{L}_{X_J} V^c + \mathcal{L}_{X_K} V^a \wedge \mathcal{L}_{X_I} \mathcal{L}_{X_L} V^b \wedge \mathcal{L}_{X_J} V^c) \\
& - \frac{1}{8} \theta^{IJ} \theta^{KL} \mathcal{L}_{X_I} (\mathcal{L}_{X_K} V^a \wedge V^b + V^a \wedge \mathcal{L}_{X_K} V^b) \wedge \mathcal{L}_{X_J} \mathcal{L}_{X_L} V^c
\end{aligned} \tag{7.8}$$

Note: Lie derivatives in the action act on forms in a diffeomorphic invariant way. Hence the action is diffeomorphisms invariant. However, the action of \mathcal{L}_{X_I} on a Lorentz tensor is *not* Lorentz covariant, since for example, using the spin connection ω^{ab} with vanishing torsion,

$$\mathcal{L}_{X_I} V^a = DX_I^a + (i_{X_I} \omega^{ab}) V^b \tag{7.9}$$

with $X_I^a = X_I^\mu V_\mu^a$, $DX_I^a = dX_I^a - \omega^{ab} X_I^b$. This expression explicitly contains a non-Lorentz covariant term due to the “naked” connection ω^{ab} . Only with successive integrations by parts one recovers the manifest local Lorentz invariance of the action (that we recall is guaranteed by invariance of the noncommutative action under noncommutative local Lorentz transformations and by the SW map construction). Thus, if written in terms of the form components, only *bona fide* Lorentz tensors appear in the action, with all indices contracted to yield a scalar.

8 Conclusions

The fields and the action of the NC vielbein gravity (+ fermions) constructed in [10] have been expanded via the SW map to second order in the noncommutativity parameter θ . The expanded action involves only the classical (commuting) fields of usual gravity coupled to fermions, and the background commuting vector fields X_I that define the abelian twist. The action is *real*, thanks to the compatibility of the SW map with the hermiticity conditions on

the field; it is also charge conjugation invariant due to the compatibility of the SW map with the charge conjugation conditions on the fields. This implies that the bosonic action is *even in* θ . In Appendix B these compatibilities are shown in general, without reference to the specific model considered in this paper.

The expanded action is invariant under usual diffeomorphisms and local Lorentz transformations.

In its use in a geometric theory, we found convenient to reformulate the SW map in the geometric language of exterior forms: this allowed to generalize the SW map to arbitrary abelian twists.

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A Twist differential geometry

The noncommutative deformation of gravity considered here and in ref. [10] relies on the existence (in the deformation quantization context, see for ex [25]) of an associative \star -product between functions and more generally an associative \wedge_\star exterior product between forms that satisfies the following properties:

- Compatibility with the undeformed exterior differential:

$$d(\tau \wedge_\star \tau') = d(\tau) \wedge_\star \tau' = \tau \wedge_\star d\tau' \quad (\text{A.1})$$

- Compatibility with the undeformed integral (graded cyclicity property):

$$\int \tau \wedge_\star \tau' = (-1)^{\deg(\tau)\deg(\tau')} \int \tau' \wedge_\star \tau \quad (\text{A.2})$$

with $\deg(\tau) + \deg(\tau') = D = \text{dimension of the spacetime manifold}$, and where here τ and τ' have compact support (otherwise stated we require (A.2) to hold up to boundary terms).

- Compatibility with the undeformed complex conjugation:

$$(\tau \wedge_\star \tau')^* = (-1)^{\deg(\tau)\deg(\tau')} \tau'^* \wedge_\star \tau^* \quad (\text{A.3})$$

Following [10] we describe here a (quite wide) class of twists whose \star -products have all these properties. As a particular case we have the Groenewold-Moyal \star -product

$$f \star g = \mu \{ e^{\frac{i}{2} \theta^{\rho\sigma} \partial_\rho \otimes \partial_\sigma} f \otimes g \}, \quad (\text{A.4})$$

where the map μ is the usual pointwise multiplication: $\mu(f \otimes g) = fg$, and $\theta^{\rho\sigma}$ is a constant antisymmetric matrix.

Abelian Twist

Let Ξ be the linear space of smooth vector fields on a smooth manifold M , and $U\Xi$ its universal enveloping algebra. A twist $\mathcal{F} \in U\Xi \otimes U\Xi$ defines the associative \star -product

$$f \star g = \mu\{\mathcal{F}^{-1}f \otimes g\} \quad (\text{A.5})$$

where the map μ is the usual pointwise multiplication: $\mu(f \otimes g) = fg$. The product associativity relies on the defining properties of the twist [8, 25].

Explicit examples of twist are provided by the so-called abelian twists:

$$\mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{IJ}X_I \otimes X_J} \quad (\text{A.6})$$

where $\{X_I\}$ is a set of mutually commuting vector fields globally defined on the manifold, and θ^{IJ} is a constant antisymmetric matrix. The corresponding \star -product is in general position dependent because the vector fields X_a are in general x -dependent. In the special case that there exists a global coordinate system on the manifold we can consider the vector fields $X_a = \frac{\partial}{\partial x^a}$. In this instance we have the Moyal twist, cf. (A.4):

$$\mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} \quad (\text{A.7})$$

Deformed exterior product

For abelian twists (A.6), the deformed exterior product between forms is defined as

$$\begin{aligned} \tau \wedge_\star \tau' &\equiv \sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n \theta^{I_1 J_1} \dots \theta^{I_n J_n} (\mathcal{L}_{X_{I_1}} \dots \mathcal{L}_{X_{I_n}} \tau) \wedge (\mathcal{L}_{X_{J_1}} \dots \mathcal{L}_{X_{J_n}} \tau') \\ &= \tau \wedge \tau' + \frac{i}{2} \theta^{IJ} (\mathcal{L}_{X_I} \tau) \wedge (\mathcal{L}_{X_J} \tau') + \frac{1}{2!} \left(\frac{i}{2}\right)^2 \theta^{I_1 J_1} \theta^{I_2 J_2} (\mathcal{L}_{X_{I_1}} \mathcal{L}_{X_{I_2}} \tau) \wedge (\mathcal{L}_{X_{J_1}} \mathcal{L}_{X_{J_2}} \tau') + \dots \end{aligned}$$

where the commuting tangent vectors X_I act on forms via the Lie derivatives \mathcal{L}_{X_I} . This product is associative, and the above formula holds also for τ or τ' being a 0-form (i.e. a function).

Exterior derivative

The exterior derivative satisfies the usual (graded) Leibniz rule, since it commutes with the Lie derivative:

$$d(f \star g) = df \star g + f \star dg \quad (\text{A.8})$$

$$d(\tau \wedge_\star \tau') = d\tau \wedge_\star \tau' + (-1)^{\deg(\tau)} \tau \wedge_\star d\tau' \quad (\text{A.9})$$

Integration: graded cyclicity

If we consider an abelian twist (A.6) given by globally defined commuting vector fields X_a , then the usual integral is cyclic under the \star -exterior products of forms, i.e., up to boundary terms,

$$\int \tau \wedge_\star \tau' = (-1)^{\deg(\tau)\deg(\tau')} \int \tau' \wedge_\star \tau \quad (\text{A.10})$$

with $\deg(\tau) + \deg(\tau') = D = \text{dimension of the spacetime manifold}$. In fact we have, up to boundary terms,

$$\int \tau \wedge_\star \tau' = \int \tau \wedge \tau' = (-1)^{\deg(\tau)\deg(\tau')} \int \tau' \wedge \tau = (-1)^{\deg(\tau)\deg(\tau')} \int \tau' \wedge_\star \tau \quad (\text{A.11})$$

For example at first order in θ ,

$$\int \tau \wedge_\star \tau' = \int \tau \wedge \tau' - \frac{i}{2} \theta^{ab} \int \mathcal{L}_{X_a}(\tau \wedge \mathcal{L}_{X_b} \tau') = \int \tau \wedge \tau' - \frac{i}{2} \theta^{ab} \int di_{X_a}(\tau \wedge \mathcal{L}_{X_b} \tau') \quad (\text{A.12})$$

where we used the Cartan formula $\mathcal{L}_{X_a} = di_{X_a} + i_{X_a}d$.

Complex conjugation

If we choose real fields X_a in the definition of the twist (A.6), it is immediate to verify that:

$$(f \star g)^* = g^* \star f^* \quad (\text{A.13})$$

$$(\tau \wedge_\star \tau')^* = (-1)^{\deg(\tau)\deg(\tau')} \tau'^* \wedge_\star \tau^* \quad (\text{A.14})$$

since sending i into $-i$ in the twist (A.7) amounts to send θ^{ab} into $-\theta^{ab} = \theta^{ba}$, i.e. to exchange the order of the factors in the \star -product.

B Seiberg Witten map: hermiticity and charge conjugation properties

We here sharpen the results of [18] concerning the general properties of hermiticity and reality of SW map (in particular in [18] the gauge group was an internal group, here it can also be the Lorentz group, for example in its spin representation). Applying these general properties we derive the hermiticity and charge conjugation properties of the fields of NC gravity coupled to spinors used in Section 2.5.

Given a representation $\rho : G \rightarrow GL(n, C)$ of a group G we can consider three other representations of G : i) the inverse Hermitian representation, defined for all $g \in G$ by $\rho'(g) = \rho(g)^{-1\dagger}$ (without inversion we would not have $\rho'(g\tilde{g}) = \rho'(g)\rho'(\tilde{g})$); ii) the complex conjugate representation, $\rho^*(g) = \rho(g)^*$ where $*$ denotes complex conjugation, iii) the inverse transpose representation $\rho^T(g) = \rho(g)^{-1T}$ (i.e. the complex conjugate representation of ρ'). The representation ρ is unitary if $\rho' = \rho$.

These representations induce representations of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$. If $g = e^{i\lambda_a T^a}$ with $\lambda_a \in R$, we find

$$\rho'(T^a) = [\rho(T^a)]^\dagger, \quad \rho^*(T^a) = -[\rho(T^a)]^*, \quad \rho^T(T^a) = -[\rho(T^a)]^T. \quad (\text{B.15})$$

In turn these Lie algebra representations can be extended to representations of the universal enveloping algebra $U\mathfrak{g}$ of \mathfrak{g} by linearity and multiplicativity (we recall that $U\mathfrak{g}$ is the associative

algebra of polynomials of elements of \mathfrak{g} , where the commutator $TT' - T'T$ is identified with the Lie bracket $[T, T']$.

From (B.15) we have

$$\rho'(A) = \rho(A)^\dagger, \quad \rho^*(A) = -\rho(A)^*, \quad \rho^T(A) = -\rho(A)^T.$$

We show that these relations hold also for the NC fields. In other words the SW map is compatible with hermitian conjugation, with complex conjugation and with transposition:

$$\widehat{\rho'(A)} = \widehat{\rho(A)}^\dagger, \quad \widehat{\rho'(\varepsilon)} = \widehat{\rho(\varepsilon)}^\dagger \quad (\text{B.16})$$

i.e., $(\widehat{\rho(A)})^\dagger = \widehat{\rho(A)}^\dagger$, $(\widehat{\rho(\varepsilon)})^\dagger = \widehat{\rho(\varepsilon)}^\dagger$, that for short we rewrite $\widehat{A}^\dagger = \widehat{A}^\dagger$, $\widehat{\varepsilon}^\dagger = \widehat{\varepsilon}^\dagger$,

$$\widetilde{\rho^*(A)} = -\widehat{\rho(A)}^* \quad , \quad \widetilde{\rho^*(\varepsilon)} = -\widehat{\rho(\varepsilon)}^* \quad , \quad (\text{B.17})$$

and

$$\widetilde{\rho^T(A)} = -\widehat{\rho(A)}^T \quad , \quad \widetilde{\rho^T(\varepsilon)} = -\widehat{\rho(\varepsilon)}^T \quad , \quad (\text{B.18})$$

where $\widetilde{}$ denotes the SW map with $-\theta$ noncommutativity. More explicitly formula (B.18) reads

$$SW[\rho^T(A), -\theta] = -SW[\rho(A), \theta]^T, \quad SW[\rho^T(\varepsilon)\rho^T(A), -\theta] = -SW[\rho(\varepsilon), \rho(A), \theta]^T;$$

and similarly for formulae (B.17) and (B.16).

Proof of (B.16), (B.17) and (B.18). We recall that for generic space-time dependent matrices M and N , under complex conjugation, transposition and hermitian conjugation we have

$$(M \star N)^* = M^* \star_{-\theta} N^* \quad , \quad (M \star N)^T = N^T \star_{-\theta} M^T \quad , \quad (M \star N)^\dagger = N^\dagger \star M^\dagger \quad . \quad (\text{B.19})$$

The Hermitian conjugates of the relations (3.56), (3.57) in the representation ρ , are

$$\widehat{\rho(A_\mu)}^{n+1\dagger} = -\frac{1}{4(n+1)}\theta^{\rho\sigma}\{\widehat{\rho(A_\rho)}^\dagger, \partial_\sigma \widehat{\rho(A_\mu)}^\dagger + \widehat{\rho(F_{\sigma\mu})}^\dagger\}_\star^n \quad (\text{B.20})$$

$$\widehat{\rho(\varepsilon)}^{n+1\dagger} = -\frac{1}{4(n+1)}\theta^{\rho\sigma}\{\widehat{\rho(A_\rho)}^\dagger, \partial_\sigma \widehat{\rho(\varepsilon)}^\dagger\}_\star^n; \quad (\text{B.21})$$

where by definition $\widehat{\rho(F_{\nu\rho})} \equiv \partial_\nu(\widehat{\rho(A_\rho)}) - \partial_\rho(\widehat{\rho(A_\nu)}) - i\widehat{\rho(A_\nu)} \star \widehat{\rho(A_\rho)} + i\widehat{\rho(A_\rho)} \star \widehat{\rho(A_\nu)}$. Relations (3.56), (3.57) in the representation ρ' , are

$$\widehat{\rho'(A_\mu)}^{n+1} = -\frac{1}{4(n+1)}\theta^{\rho\sigma}\{\widehat{\rho'(A_\rho)}, \partial_\sigma \widehat{\rho'(A_\mu)} + \widehat{\rho'(F_{\sigma\mu})}\}_\star^n \quad (\text{B.22})$$

$$\widehat{\rho'(\varepsilon)}^{n+1} = -\frac{1}{4(n+1)}\theta^{\rho\sigma}\{\widehat{\rho'(A_\rho)}, \partial_\sigma \widehat{\rho'(\varepsilon)}\}_\star^n. \quad (\text{B.23})$$

Since these two sets of relations have the same structure in terms of their respective variables $\widehat{\rho(A)}^\dagger, \widehat{\rho(\varepsilon)}^\dagger$ and $\widehat{\rho'(A)}, \widehat{\rho'(\varepsilon)}$, and since at zeroth order in θ , $\rho(A)^\dagger = \rho'(A)$, $\rho(\varepsilon)^\dagger = \rho'(\varepsilon)$ the compatibility (B.16) is iteratively proven at all orders in θ .

We proceed similarly in order to prove (B.17); the complex conjugate of the relations (3.56), (3.57) in the representation ρ can be written as

$$-\widehat{\rho(A_\mu)}^{n+1*} = \frac{1}{4(n+1)}\theta^{\rho\sigma}\{-\widehat{\rho(A_\rho)}^*, -\partial_\sigma\widehat{\rho(A_\mu)}^* - \widehat{\rho(F_{\sigma\mu})}^*\}_{\star_{-\theta}}^n \quad (\text{B.24})$$

$$-\widehat{\rho(\varepsilon)}^{n+1*} = \frac{1}{4(n+1)}\theta^{\rho\sigma}\{-\widehat{\rho(A_\rho)}^*, -\partial_\sigma\widehat{\rho(\varepsilon)}^*\}_{\star_{-\theta}}^n. \quad (\text{B.25})$$

Relations (3.56), (3.57) in the representation ρ^* and with noncommutativity parameter $-\theta$ (and corresponding star product $\star_{-\theta}$) read

$$\widetilde{\rho^*(A_\mu)}^{n+1} = \frac{1}{4(n+1)}\theta^{\rho\sigma}\{\widetilde{\rho^*(A_\rho)}, \partial_\sigma\widetilde{\rho^*(A_\mu)} + \widetilde{\rho^*(F_{\sigma\mu})}\}_{\star_{-\theta}}^n \quad (\text{B.26})$$

$$\widetilde{\rho^*(\varepsilon)}^{n+1} = \frac{1}{4(n+1)}\theta^{\rho\sigma}\{\widetilde{\rho^*(A_\rho)}, \partial_\sigma\widetilde{\rho^*(\varepsilon)}\}_{\star_{-\theta}}^n. \quad (\text{B.27})$$

Since these two sets of relations have the same structure in terms of their respective variables $-\widehat{\rho(A)}^*$, $-\widehat{\rho(\varepsilon)}^*$ and $\widetilde{\rho^*(A)}$, $\widetilde{\rho^*(\varepsilon)}$, and since at zeroth order in θ , $-\rho(A)^* = \rho^*(A)$, $-\rho(\varepsilon)^* = \rho^*(\varepsilon)$, the compatibility (B.16) is iteratively proven at all orders in θ .

From (B.16) and (B.17) easily follows the compatibility of the SW map with transposition, eq. (B.18).

A similar iterative procedure shows that for matter fields transforming in the adjoint (see (3.63) and (4.70)), given $\rho'(\phi) = \rho(\phi)^\dagger$, $\rho^*(\phi) = -\rho(\phi)^*$ and $\rho^T(\phi) = -\rho(\phi)^T$, we have $SW[\rho'(\phi), \rho'(A), \theta] = SW[\rho(\phi), \rho(A), \theta]^\dagger$ and

$$SW[\rho^*(\phi), \rho^*(A), -\theta] = -SW[\rho(\phi), \rho(A), \theta]^*, \quad SW[\rho^T(\phi), \rho^T(A), -\theta] = -SW[\rho(\phi), \rho(A), \theta]^T,$$

that we rewrite as

$$\widehat{\rho(\phi)'} = \widehat{\rho(\phi)}^\dagger, \quad \widetilde{\rho^*(\phi)} = -\widehat{\rho(\phi)}^*, \quad \widetilde{\rho^T(\phi)} = -\widehat{\rho(\phi)}^T; \quad (\text{B.28})$$

with abuse of notations we simply write the first of these relations $\widehat{\phi}^\dagger = \widehat{\phi}^\dagger$.

Finally the same iterative procedure shows that for matter field in the fundamental (see (3.62) and (4.69)), given $\rho^*(\phi) = \rho(\phi)^*$, we have $SW[\rho^*(\phi), \rho^*(A), -\theta] = SW[\rho(\phi), \rho(A), \theta]^*$, i.e.,

$$\widetilde{\phi^*} = \widehat{\phi}^*. \quad (\text{B.29})$$

Charge conjugation is the transformation that maps a particle representation of a symmetry group to its complex conjugate representation ($\rho \rightarrow \rho^*$), hence the compatibility of SW map with complex conjugation is the compatibility with charge conjugation. On spinor fields complex conjugation and charge conjugation differ by a unitary transformation (see (B.35), (B.37)). As we show below also in this case the SW map is compatibe with charge conjugation.

Spin representation of Lorentz group

We now consider the gauge group $G = SL(2, C)$ and fix the representation ρ (that from now

on we omit writing) to be the one determined by the 4×4 matrices in the spinor representation used throughout the paper (the $(\frac{1}{2}, \frac{1}{2})$ -spinor representation). We prove that $\hat{\Omega} = i\hat{A}$ satisfies the hermiticity and symmetry conditions,

$$\hat{\Omega}^\dagger = -\gamma_0 \hat{\Omega} \gamma_0 \quad , \quad \hat{\Omega}^T = C \hat{\Omega} C ; \quad (\text{B.30})$$

these are precisely the conditions (2.28) and (2.30), that we have here rewritten using hatted variables in order to stress that these conditions are now *derived* from the hermiticity and symmetry properties of the classical fields and of the SW map.

We know that the classical fields satisfy $\Omega^\dagger = -\gamma_0 \Omega \gamma_0$, $\Omega^T = C \Omega C$, i.e., since $A = -i\Omega$, $\gamma_0 = \gamma_0^{-1}$, $C = -C^{-1}$, we know that $A^\dagger = \gamma_0 A \gamma_0^{-1}$, $-A^T = C A C^{-1}$. We then have

$$\hat{A}^\dagger = \widehat{A^\dagger} = \widehat{\gamma_0 A \gamma_0^{-1}} = \gamma_0 \hat{A} \gamma_0^{-1} , \quad (\text{B.31})$$

$$\hat{A}^T = -\widetilde{\rho^T(A)} = -(\widetilde{-A^T}) = -\widetilde{C A C^{-1}} = -C \widetilde{A} C^{-1} , \quad (\text{B.32})$$

where we used (B.17) and (B.18), and in the last equality of each expression we used compatibility of SW map with the similarity transformations $A \rightarrow \gamma_0 A \gamma_0^{-1}$ and $A \rightarrow C A C^{-1}$ respectively (for example $SW[C A C^{-1}, \theta] = C SW[A, \theta] C^{-1}$). These expressions immediately imply (B.30).

A similar proof shows that the gauge parameter $\hat{\epsilon}$, as well as the vielbein \hat{V} satisfy the hermiticity and symmetry conditions,

$$\hat{\epsilon}^\dagger = -\gamma_0 \hat{\epsilon} \gamma_0 \quad , \quad \hat{\epsilon}^T = C \hat{\epsilon} C ; \quad (\text{B.33})$$

$$\hat{V}^\dagger = \gamma_0 \hat{V} \gamma_0 \quad , \quad \hat{V}^T = C \hat{V} C ; \quad (\text{B.34})$$

hint: use that $\hat{\epsilon} = i\hat{\varepsilon}$, where $\hat{\epsilon}$ corresponds to the gauge potential $\hat{\Omega}$, and $\hat{\varepsilon}$ corresponds to the gauge potential \hat{A} (see (2.24) and (3.49)). For (B.34) use (B.28). Relations (B.34) coincide with the reality and charge conjugation conditions considered in (2.28) and (2.30).

Charge Conjugation

Under charge conjugation we have

$$\psi \rightarrow \psi^C = C(\bar{\psi})^T = -\gamma_0 C \psi^* , \quad V \rightarrow V^C = V , \quad A \rightarrow A^C = A \quad (\text{B.35})$$

Applying the SW map we obtain the noncommutative fields relations $\widehat{V^C} = \hat{V}$, $\widehat{A^C} = \hat{A}$, that with the help of (B.34) and (B.31) equivalently read

$$\widetilde{V^C} = C \hat{V}^T C , \quad \widetilde{A^C} = C \hat{A}^T C . \quad (\text{B.36})$$

The charge conjugation operation on the gauge potential can also be written

$$A \rightarrow A^C = (-\gamma_0 C) \rho^*(A) (-\gamma_0 C)^{-1} = -(-\gamma_0 C) A^* (-\gamma_0 C)^{-1} . \quad (\text{B.37})$$

Then

$$\begin{aligned} \widetilde{\psi^C} &= SW[\psi^C, A^C, -\theta] = SW[-\gamma_0 C \psi^*, Ad_{-\gamma_0 C} \rho^*(A), -\theta] = -\gamma_0 C SW[\psi^*, \rho^*(A), -\theta] \\ &= -\gamma_0 C \widetilde{\psi^*} = -\gamma_0 C \hat{\psi}^* \end{aligned} \quad (\text{B.38})$$

where in the last passage we used (B.29).

In the previous sections we used the notation $\hat{\psi} = \hat{\psi}(\psi, \Omega, \theta)$ to denote the SW map $\hat{\psi} = SW[\psi, A, \theta]$, and similarly for \hat{V} and $\hat{\Omega} = i\hat{A}$. Then relations (B.36) and (B.38) coincide with equations (6.2).

C Gamma matrices in $D = 4$

We summarize in this Appendix our gamma matrix conventions in $D = 4$.

$$\eta_{ab} = (1, -1, -1, -1), \quad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \quad [\gamma_a, \gamma_b] = 2\gamma_{ab}, \quad (C.1)$$

$$\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_5\gamma_5 = 1, \quad \varepsilon_{0123} = -\varepsilon^{0123} = 1, \quad (C.2)$$

$$\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0, \quad \gamma_5^\dagger = \gamma_5 \quad (C.3)$$

$$\gamma_a^T = -C\gamma_aC^{-1}, \quad \gamma_5^T = C\gamma_5C^{-1}, \quad C^2 = -1, \quad C^\dagger = C^T = -C \quad (C.4)$$

C.1 Useful identities

$$\gamma_a\gamma_b = \gamma_{ab} + \eta_{ab} \quad (C.5)$$

$$\gamma_{ab}\gamma_5 = \frac{i}{2}\epsilon_{abcd}\gamma^{cd} \quad (C.6)$$

$$\gamma_{ab}\gamma_c = \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abcd}\gamma_5\gamma^d \quad (C.7)$$

$$\gamma_c\gamma_{ab} = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a - i\varepsilon_{abcd}\gamma_5\gamma^d \quad (C.8)$$

$$\gamma_a\gamma_b\gamma_c = \eta_{ab}\gamma_c + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abcd}\gamma_5\gamma^d \quad (C.9)$$

$$\gamma^{ab}\gamma_{cd} = -i\varepsilon^{ab}_{cd}\gamma_5 - 4\delta_{[c}^{[a}\gamma^{b]}_{d]} - 2\delta_{cd}^{ab} \quad (C.10)$$

where $\delta_{cd}^{ab} = \frac{1}{2}(\delta_c^a\delta_d^b - \delta_c^b\delta_d^a)$ and indices antisymmetrization in square brackets has total weight 1.

C.2 Charge conjugation and Majorana condition

$$\text{Dirac conjugate } \bar{\psi} \equiv \psi^\dagger\gamma_0 \quad (C.11)$$

$$\text{Charge conjugate spinor } \psi^C = C(\bar{\psi})^T \quad (C.12)$$

$$\text{Majorana spinor } \psi^C = \psi \Rightarrow \bar{\psi} = \psi^TC \quad (C.13)$$

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